

STABLE MINIMAL GRAPHS IN THE HEISENBERG GROUP \mathbb{H}^n

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ABSTRACT. We prove that a strictly stable minimal C_h^2 intrinsic graph G is locally area-minimizing, i.e. given any C_h^1 graph S with the same boundary, $\text{Area}(G) < \text{Area}(S)$ unless $G = S$. As a consequence we show the existence and the uniqueness of C^∞ minimal graphs with prescribed small boundary datum.

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1. INTRODUCTION

The Heisenberg group, denoted as \mathbb{H}^n can be identified with \mathbb{R}^{2n+1} , with the choice at every point of an *horizontal $2n$ -dimensional distribution* \mathcal{H} , such that $[\mathcal{H}, \mathcal{H}] = \mathcal{H}_2$ has dimension 1,

(1.1) $T\mathbb{H}^n = \mathcal{H} \oplus \mathcal{H}_2$, $[\mathcal{H}, \mathcal{H}] = \mathcal{H}_2$ and $[\mathcal{H}, \mathcal{H}_0] = 0$.

Starting from the seminal paper [21] by Garofalo and Nhieu, where general properties and existence of sets with minimum perimeter are proved in Carnot groups, a remarkable effort has been devoted to the development of an exhaustive theory for minimizers of the area functional in the sub-Riemannian setting. However the theory is still very far from being complete.

Due to the lack of symmetry of the space, surfaces can have very different expression if expressed as graph on different variables. Identifying via the exponential coordinates the Heisenberg group with its algebra, we can consider graphs defined on the horizontal distribution \mathcal{H} with values in \mathcal{H}_2 . Graphs of this type with prescribed mean curvature f , called t -graphs, are critical points of the functional

$$\mathcal{F}(u) = \int_{\Omega} |\nabla u + \vec{F}| + \int_{\Omega} f u,$$

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on a domain Ω , where \vec{F} is a vector field and $f \in L^\infty(\Omega)$. In [9] Cheng et al. proved the existence of C^1 minimizing t -graphs in \mathbb{H}^n that at our knowledge, this is the only existence results for prescribed mean curvature graphs.

On the other side while choosing graphs whose normal at every point belongs to the horizontal distribution we obtain the so called intrinsic graphs introduced by Franchi, Serapioni and Serra Cassano in [14] and [15]. It has been proved, in [1] and [11], that in this case there are non-linear vector fields

$$(X_{1,u}, \dots, X_{n,u})$$

such that the area functional of the graph of u can be written as

$$A(u) = \int_{\Omega} \sqrt{1 + |\nabla^u u|^2} \, d\mathcal{L}^{2n}.$$

We remark that this sub-Riemannian area functional is not convex. This is why standard existence technique do not apply. However existence of minima of this functional have been obtained by Serra Cassano and Vittone, [30], but they have only BV regularity. In particular it is not clear if they satisfy the associated first variation equation which express the curvature of a surface:

$$(1.2) \quad H(u) = dA(u) = \sum_{i=1}^{2n-1} X_i^u \left(\frac{X_i^u(u)}{\sqrt{1 + |\nabla^u(u)|^2}} \right) = 0.$$

This is a second order quasilinear subelliptic equation, but due to the expression of the vector fields $X_{i,u}$, the regularity of the solution is not understood, and there is still a big gap between this existence result for minima and their regularity properties. Regularity results in the three-dimensional case, with special attention to the first Heisenberg group \mathbb{H}^1 has been obtained only for Lipschitz continuous solutions. Among them, we would like to stress [10], [7], [2], [19], [20] and [17]. In dimension bigger then three the only regularity result in \mathbb{H}^n , $n > 1$, is proved by Capogna et al. in [8], always for Lipschitz continuous solutions. The main obstacle to study directly the the problem

$$(1.3) \quad \begin{cases} H_v = 0, & \text{in } \Omega \\ v = \phi, & \text{in } \partial\Omega \end{cases}$$

is the fact that the operator

$$(1.4) \quad L_u(v) = dH(u)(v)$$

does not satisfy the maximum principle for general functions u . Indeed in the subriemannian setting the stability operator $L_u v$ takes the form

$$\begin{aligned} L_u v &= \sum_{i,j=1}^{2n-1} X_i^u (a_{ij}(\nabla^u u) X_j^u v) + \sum_{i=1}^{2n-1} (\partial_{2n} u) a_{i2n-1}(\nabla^u u) X_i^u v \\ &\quad + \left(\partial_{2n} \left(\frac{X_{2n-1}^u u}{\sqrt{1 + |\nabla^u u|^2}} \right) + \sum_{i=1}^{2n-1} X_i^u (\partial_{2n} u a_{i2n-1}(\nabla^u u)) \right) v, \end{aligned}$$

(see Proposition 3.1), and nothing is known on the sign of the zero order term. In addition Schauder estimates are not available for general subriemannian vector fields.

In this paper we investigate properties of strictly stable critical points, which are graphs G_u with vanishing mean curvature in an open set Ω such that, for all compact test functions $v \in C_h^2(\Omega)$, $v \not\equiv 0$, the *index form*

$$\mathcal{J}(v, v) = - \langle L_u(v), v \rangle$$

is strictly positive, where $L_u(v)$ is defined in (1.4). Our main result is stated in Theorem 5.3 below:

Let $u \in C_h^2(\tilde{\Omega})$ and let u be a strictly stable critical point of the area functional in a domain $\Omega \subset \tilde{\Omega}$. Then there exists a tubular neighborhood U of Ω such that for any C_h^1 graph $S \subset U$, $\partial\Omega = \partial S$, we have $A(G_u) < A(S)$ or $G_u = S$.

This result extends to the present setting the one proved in Riemannian manifolds by White [31], using regularity theorems from geometric measure theory. Grosse-Brauckmann [23], see also [28, §109], show that another way to prove this result is to foliate a neighbourhood of a strictly stable extremal with stationary surfaces. The field of normals to the leaves is a calibration, and the statement follows by the divergence theorem.

When u is a strictly stable critical point u , $L_u(v)$ is positive, hence invertible. Hence it is possible to show an existence result in a neighborhood of any stable point. In particular, since $u = 0$ has this property we deduce that

There exists $\varepsilon_0 > 0$ such that if $\|\phi\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$, then the problem (1.3) has an unique solution $v \in C^\infty(\Omega)$.

The delicate aspect of the proof of the existence theorem is the lack of Schauder estimates at the boundary. As we mentioned before, they are not known in the subriemannian setting and the only result in this direction is due to Jerison [24]. However his proof can not be repeated in general Lie groups, since it is based on Fourier transform. On the contrary, internal Schauder estimates are well known in this context, after the results of [29] (we also quote a more recent contribution [6]). Hence we prove an ad hoc version of Schauder estimates with a penalization on the boundary of the set, which are sufficient to obtain the result.

Using this existence result, we are able to follow the same idea of [23] and construct a foliation by minimal graphs in a tubular neighborhood of a strictly stable minimal graph. The positivity of the operator L_u for strictly stable function u implies that the operator L_u satisfies the maximum principle (see also Proposition 5.1). As a consequence we will establish the proof of Theorem 5.3.

As a corollary we will deduce that this critical point of the area functional, found above for $\|\phi\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$, is indeed a stable minimum with prescribed boundary datum.

The paper is organized as follows. In Section 2 we provide the necessary background on the sub-Riemannian Heisenberg group and intrinsic graphs with prescribed mean curvature. In Section 3 we introduce the stability operator. In section 3 we prove the Schauder estimates and our existence result. The local area-minimizing property of a strictly stable minimal intrinsic graph will appear in Section 5.

2. PRELIMINARIES

In this section we gather some results to be used in later sections.

2.1. The Heisenberg group \mathbb{H}^n . The structure of the Heisenberg group \mathbb{H}^n can be modeled on \mathbb{R}^{2n+1} using the following basis of left-invariant vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i}, i = 1, \dots, n-1, & X_i &= \frac{\partial}{\partial x_i} - x_{i-n+1} \frac{\partial}{\partial x_{2n}}, i = n, \dots, 2n-2, \\ X_z &= \frac{\partial}{\partial z}, & X_{2n-1} &= \frac{\partial}{\partial x_{2n-1}} - z \frac{\partial}{\partial x_{2n}}, & X_{2n} &= \frac{\partial}{\partial x_{2n}}. \end{aligned}$$

The vector fields $\{X_z, X_1, \dots, X_{2n-1}\}$ generate the *horizontal distribution* \mathcal{H} , while T is called the Reeb vector field and it is transverse to \mathcal{H} . A vector field X is called *horizontal* if $X \in \mathcal{H}$. A horizontal curve is a C^1 curve whose tangent vector lies in the horizontal distribution.

Note that

$$[X_z, X_{2n-1}] = [X_i, X_{i+n-1}] = T, \quad i = 1, \dots, n-1,$$

while the other commutators vanish, so that \mathcal{H} is a bracket-generating distribution and \mathbb{H}^n has vanishing pseudo-hermitian Webster curvature and pseudo-hermitian torsion, see [13] or [16]. For this reason the Heisenberg group is the model example of pseudo-hermitian manifolds and play the same role that the Euclidean space has with respect to a Riemannian manifold.

2.2. The left invariant metric. We shall consider on \mathbb{H}^n the Riemannian metric $g = \langle \cdot, \cdot \rangle$ so that $\{X_z, X_1, \dots, X_{2n-1}, T\}$ is an orthonormal basis at every point. The restriction of g to \mathcal{H} coincides with the usual sub-Riemannian metric in \mathbb{H}^n induced by the vector fields $X_z, X_1, \dots, X_{2n-1}$.

For any tangent vector U on \mathbb{H}^n we define $J(U) = D_U T$, where D is the Levi-Civita connection associated to the Riemannian metric g . Then we have

$$J(X_z) = X_{2n-1}, J(X_{2n-1}) = -X_z, J(X_i) = X_{i+n-1}, J(X_{i+n-1}) = -X_i,$$

for $i = 1, \dots, n-1$, and $J(T) = 0$, so that $J^2 = -\text{Id}$ when restricted to \mathcal{H} . The involution $J : \mathcal{H} \rightarrow \mathcal{H}$ provides a complex structure on \mathbb{H}^n , see [4].

2.3. Geometry of surfaces in \mathbb{H}^n . Given a C^1 surface Σ immersed in \mathbb{H}^n we define the *sub-Riemannian area* of Σ by

$$(2.1) \quad A(\Sigma) = \int_{\Sigma} |N_h| d\Sigma,$$

where N is the unit normal vector with respect to the metric g , N_h is the orthogonal projection of N to \mathcal{H} and $d\Sigma$ is the Riemannian area element of Σ . The singular set Σ_0 consists of those points p where \mathcal{H}_p coincides with the tangent plane $T_p \Sigma$ of Σ . We define the *horizontal unit normal vector* $v_h(p)$ and the *characteristic vector field* $Z(p)$ by

$$(2.2) \quad v_h(p) := \frac{N_h(p)}{|N_h(p)|}, \quad Z(p) := J(v_h)(p)$$

for all $p \in \Sigma - \Sigma_0$. Since Z_p is orthogonal to v_h and horizontal, we get that Z_p is tangent to Σ .

2.4. Euclidean Lipschitz graphs in \mathbb{H}^n . Let $W = \{(0, x) \in \mathbb{H}^n : x \in \mathbb{R}^{2n}\}$, we consider the graph $\Sigma = \{(z, x) : z = u(x), x \in \Omega\} \subset \mathbb{H}^n$ of an Euclidean Lipschitz function $u : \Omega \subset W \rightarrow \mathbb{R}$. Let $\{E_i\}_{i=1, \dots, 2n-1}$ be a basis of the horizontal tangent space $T\Sigma \cap \mathcal{H}$, where

$$E_i = (X_i u) X_z + X_i, \quad i = 1, \dots, 2n-1.$$

We denote by $\{X_i^u\}_{i=1,\dots,2n-1}$ a basis of the horizontal tangent vectors projected to Ω

$$\begin{aligned} X_i^u &= X_i, \quad i = 1, \dots, 2n-2 \\ X_{2n-1}^u &= \frac{\partial}{\partial x_{2n-1}} + u(x) \frac{\partial}{\partial x_{2n}}. \end{aligned}$$

The gradient ∇^u is defined as

$$\nabla^u = (X_1^u, \dots, X_{2n-1}^u)$$

and $\nabla^u u$ is well-defined and continuous in Ω , since G_u is a so-called *intrinsic graph*, [15] and [3], in fact the integral curves of X_x starting from Ω meet G_u in exactly one point. The area formula (2.1) for a graph G_u can be expressed as

$$(2.3) \quad A(G_u) = \int_{\Omega} (1 + |\nabla^u u|^2)^{1/2} d\mathcal{L}^{2n},$$

where \mathcal{L}^{2n} denotes the Lebesgue measure on Ω , [1, Proposition 2.22].

2.5. Graphs with prescribed mean curvature. Let G_u the graph of an Euclidean Lipschitz function $u : \Omega \subset W \rightarrow \mathbb{R}$. G_u has *prescribed mean curvature* f if it is a critical point of the functional

$$(2.4) \quad A(G_u \cap B) - \int_{E_u \cap B} f,$$

for any bounded open set B in the cylinder $\{(z, x) \in \mathbb{H}^n : x \in \Omega\}$. Here we have denoted by $E_u = \{(z, x) \in \mathbb{H}^n : x \in \Omega, z < u(x)\}$ the subgraph of G_u . We remark that our definition is the counterpart of the ones given in three-dimensional sub-Riemannian contact manifolds, [20] and [17], and in the Euclidean setting, [25, (12.32) and Remark 17.11].

2.6. Distance generated by vector fields. Note that the vector fields X_1^u, \dots, X_{2n-1}^u satisfy Hörmander's finite rank condition in \mathbb{R}^{2n} . Consequently they give rise to a control distance d_u , whose metric balls B_r of radius r have volume comparable to r^{2n+1} , where $2n+1$ is the homogeneous dimension of the space (\mathbb{R}^{2n}, d_u) . The distance d_u coincides with the sub-Riemannian distance restricted to the vertical plane containing Ω , see [27].

2.7. Function spaces. Let f be a continuous function on $E \subset \mathbb{H}^n$. We say that $f \in C_h^k(E)$ if $Y_1(\dots(Y_k(f)))$ exists continuous, for any $Y_1, \dots, Y_k \in \mathcal{H}$. It is easy to check that $f \in C_h^{2k}(E)$ implies $f \in C^k(E)$.

Given $0 < \alpha < 1$, we define $C_d^{0,\alpha}(\Omega)$ as the space composed by all functions $f : \Omega \rightarrow \mathbb{R}$ having finite the following norm

$$\|f\|_{\alpha,d,\Omega} := \sup_{\Omega} |f| + \sup_{x,y \in \Omega, x \neq y} d_{x,y}^{\alpha} \frac{|f(x) - f(y)|}{d_u(x,y)^{\alpha}},$$

where

$$d_{x,y} := \min(d_u(x, \partial\Omega), d_u(y, \partial\Omega))$$

and d_u denotes the distance induced by the vector fields X_i^u , $i = 1, \dots, 2n-1$. We also say that $f \in C_d^{2+\alpha}(\Omega)$ if the norm

$$\|f\|_{2+\alpha,d,\Omega} := \sup_{\Omega} |f| + \sum_{i=1}^{2n-1} \sup_{\Omega} d |X_i^u f| + \sum_{i,j=1}^{2n-1} \|d^2 X_i^u X_j^u f\|_{\alpha,d,\Omega} < +\infty.$$

2.8. First variation of the Area functional. We first recall that, if u is a graph, Lipschitz with respect to the Euclidean metric, the area of the graph G_u is

$$A(G_u) = \int_{\Omega} \sqrt{1 + |\nabla^u u|^2} d\mathcal{L}^{2n}$$

(see for example [18], [12], [8]). Moreover, if u is a critical point of the functional (2.4) we have that

$$(2.5) \quad \int_{\Omega} \left\{ \frac{\sum_{i=1}^{2n-2} X_i u X_i \phi + X_{2n-1}^u u (X_{2n-1}^u)^* \phi}{\sqrt{1 + |\nabla^u u|^2}} - f \phi \right\} d\mathcal{L}^{2n} = 0$$

for any $\phi \in C_0^1(\Omega)$. In order to simplify notations we will denote

$$(2.6) \quad A_i(\nabla^u u) = \frac{X_i u}{\sqrt{1 + |\nabla^u u|^2}} \text{ for } i = 1 \cdots 2n-2 \quad A_{2n-1}(\nabla^u u) = \frac{X_{2n-1}^u u}{\sqrt{1 + |\nabla^u u|^2}}.$$

As a consequence the definition of *mean curvature* for the graph G_u reads (see for example [18], [12], [8]):

$$(2.7) \quad H_u := \sum_{i=1}^{2n-1} X_i^u \left(\frac{X_i^u u}{\sqrt{1 + |\nabla^u u|^2}} \right) = \sum_{i=1}^{2n-1} X_i^u A_i(\nabla^u u)$$

We can also consider the operator of the minimal surface equation in the non-divergence form

$$(2.8) \quad M_u = \sum_{i=1}^{2n-1} a_{ij}(\nabla^u u) X_i^u X_j^u(u),$$

where

$$(2.9) \quad a_{ij} : \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1 + |p|^2}.$$

3. STABILITY OPERATOR FOR MINIMAL GRAPHS

In this section we define the stability operator for subriemannian minimal graphs.

Proposition 3.1. *Let $G_u = \{z = u(x) : x \in \Omega\}$ be a C_h^2 graph in \mathbb{H}^n and let $v \in C_h^2(\Omega)$. We consider a variation of G_u of the form $G_{u+sv} = \{z = u(x) + sv(x) : x \in \Omega\}$. Then the second variation of the area of G_u is*

$$\left. \frac{d^2}{ds^2} \right|_{s=0} A(G_{u+sv}) = \int_{\Omega} v L_u v d\mathcal{L}^{2n},$$

where

$$(3.1) \quad \begin{aligned} L_u v = & \sum_{i,j=1}^{2n-1} X_i^u \left(\frac{a_{ij}(\nabla^u u)}{\sqrt{1 + |\nabla^u u|^2}} X_j^u v \right) + \sum_{i=1}^{2n-1} (\partial_{2n} u) a_{i2n-1}(\nabla^u u) X_i^u v \\ & + \left(\partial_{2n} \left(\frac{X_{2n-1}^u u}{\sqrt{1 + |\nabla^u u|^2}} \right) + \sum_{i=1}^{2n-1} X_i^u (\partial_{2n} u a_{i2n-1}(\nabla^u u)) \right) v, \end{aligned}$$

and the functions a_{ij} are defined in (2.9).

Proof. It is a standard computation show that

$$\begin{aligned} \frac{d}{ds} H_{u+sv} &= \sum_{i=1}^{2n-1} X_i^{u+sv} \left(\frac{X_i^{u+sv}(v)}{\sqrt{1 + |\nabla^{u+sv}(u+sv)|^2}} \right) - \sum_{i,j=1}^{2n-1} X_i^{u+sv} \left(\frac{X_i^{u+sv}(u+sv) X_j^{u+sv}(u+sv) X_j v}{(1 + |\nabla^{u+sv}(u+sv)|^2)^{3/2}} \right) \\ &\quad + v \partial_{2n} \left(\frac{X_{2n-1}^u(u+sv)}{\sqrt{1 + |\nabla^{u+sv}(u+sv)|^2}} \right) + X_{2n-1}^u \left(\frac{v \partial_{2n}(u+sv)}{\sqrt{1 + |\nabla^{u+sv}(u+sv)|^2}} \right) \\ &\quad - \sum_{i=1}^{2n-1} X_i^{u+sv} \left(\frac{X_i^{u+sv}(u+sv) X_{2n-1}^{u+sv}(u+sv) v \partial_{2n} u}{(1 + |\nabla^{u+sv}(u+sv)|^2)^{3/2}} \right). \end{aligned}$$

When we evaluate this expression at $s = 0$ to obtain the following statement:

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} H_{u+sv} &= \sum_{i=1}^{2n-1} X_i^u \left(\frac{X_i^u(v)}{\sqrt{1 + |\nabla^u(u)|^2}} \right) - \sum_{i,j=1}^{2n-1} X_i^u \left(\frac{X_i^u(u) X_j^u(u) X_j v}{(1 + |\nabla^u(u)|^2)^{3/2}} \right) \\ &\quad + v \partial_{2n} \left(\frac{X_{2n-1}^u(u)}{\sqrt{1 + |\nabla^u(u)|^2}} \right) + X_{2n-1}^u \left(\frac{v \partial_{2n}(u)}{\sqrt{1 + |\nabla^u(u)|^2}} \right) \\ &\quad - \sum_{i=1}^{2n-1} X_i^u \left(\frac{X_i^u(u) X_{2n-1}^u(u) v \partial_{2n} u}{(1 + |\nabla^u(u)|^2)^{3/2}} \right) \end{aligned}$$

Observing that

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} H_{u+sv} &= \sum_{i,j=1}^{2n-1} a_{ij}(\nabla^u u) X_i^u X_j^u v + \sum_{i=1}^{2n-1} \left(\sum_{j=1}^{2n-1} X_j^u (a_{ij}(\nabla^u u)) + \partial_{2n} u a_{i2n-1}(\nabla^u u) \right) X_i^u v \\ &\quad + \left(\partial_{2n} \left(\frac{X_{2n-1}^u u}{\sqrt{1 + |\nabla^u u|^2}} \right) + \sum_{i=1}^{2n-1} X_i^u (\partial_{2n} u a_{i2n-1}(\nabla^u u)) \right) v \\ &= \sum_{i,j=1}^{2n-1} X_i^u (a_{ij}(\nabla^u u) X_j^u v) + \sum_{i=1}^{2n-1} \partial_{2n} u a_{i2n-1}(\nabla^u u) X_i^u v \\ &\quad + \left(\partial_{2n} \left(\frac{X_{2n-1}^u u}{\sqrt{1 + |\nabla^u u|^2}} \right) + \sum_{i=1}^{2n-1} X_i^u (\partial_{2n} u a_{i2n-1}(\nabla^u u)) \right) v \end{aligned}$$

we obtain the thesis. \square

We can now introduce the following definition

Definition. We call *stability operator* L_u associated to a minimal graph G_u the operator $L_u v$ defined in equation (3.1).

We consider a Euclidean Lipschitz minimal graph G_u with vanishing mean curvature in the Heisenberg group \mathbb{H}^n and a smooth domain Ω . We say that G_u is *strictly stable* if for all $v \in C_h^2(\Omega)$ with compact support, $v \not\equiv 0$, the *index form*

$$\mathcal{J}(v, v) = - \int_{\Omega} v L_u v d\mathcal{L}^{2n}$$

is strictly positive, where L_u is the stability operator defined in (3.1).

Lemma 3.2. *Let us note that there exists a constant C such that, if u is a minimum and $\|u\|_{C^2} \leq C$, then u is strictly stable*

Proof.

$$\begin{aligned} \int v L_u v &= \sum_{i,j=1}^{2n-1} \int v X_i^u \left(\frac{a_{ij}(\nabla^u u)}{\sqrt{1+|\nabla^u u|^2}} X_j^u v \right) + \sum_{i=1}^{2n-1} \int v \partial_{2n} u a_{i2n-1}(\nabla^u u) X_i^u v \\ &\quad + \int v \left(\partial_{2n} \left(\frac{X_{2n-1}^u u}{\sqrt{1+|\nabla^u u|^2}} \right) + \sum_{i=1}^{2n-1} X_i^u (\partial_{2n} u a_{i2n-1}(\nabla^u u)) \right) v \end{aligned}$$

Integrating by part the first term, and using the fact that $[X_1, X_{n+1}] = \partial_{2n}$, we get

$$\begin{aligned} \int v L_u v &= - \sum_{i,j=1}^{2n-1} \int X_i^u v \frac{a_{ij}(\nabla^u u)}{\sqrt{1+|\nabla^u u|^2}} X_j^u v - \int v \partial_{2n} u \frac{a_{ij}(\nabla^u u)}{\sqrt{1+|\nabla^u u|^2}} X_j^u v \\ &\quad + \sum_{i=1}^{2n-1} \int v \partial_{2n} u a_{i2n-1}(\nabla^u u) X_i^u v \\ &\quad - \int X_{n+1} \left(\frac{X_{2n-1}^u u}{\sqrt{1+|\nabla^u u|^2}} \right) v X_1 v + \int X_1 \left(\frac{X_{2n-1}^u u}{\sqrt{1+|\nabla^u u|^2}} \right) v X_{n+1} v \\ &\quad - \sum_{i=1}^{2n-1} \int \partial_{2n} u a_{i2n-1}(\nabla^u u) X_i^u v v - \int (\partial_{2n} u)^2 a_{i2n-1}(\nabla^u u) v^2 \end{aligned}$$

By the structure of the equation we have

$$\sum_{i,j=1}^{2n-1} \int X_i^u v \frac{a_{ij}(\nabla^u u)}{\sqrt{1+|\nabla^u u|^2}} X_j^u v \geq \int |X_i^u v|^2 \geq \int |v|^2$$

By Hölder inequality and Sobolev embedding Theorem we have

$$\int v X_i^u v \leq \int |X_i^u v| |v| \leq C_S \int |X_i^u v|^2$$

so that

$$- \int v L_u v \geq (1 - C_S \|u\|_{C^2}^2) \int |X_i^u v|^2.$$

It follows that $-\int v L_u v$ is strictly positive if $\|u\|_{C^2}^2$ is sufficiently small. \square

4. AN EXISTENCE RESULT

In this section we prove a first existence result for solutions of the problem (1.3). Precisely we show that

Proposition 4.1. *Let $u \in C_h^2(\tilde{\Omega})$ and let G_u be critical point of the area functional in \mathbb{H}^n , $n > 1$. We assume that G_u is strictly stable in a domain $\Omega \subset \tilde{\Omega}$. Then there exist $\varepsilon_0 > 0$, such that if $\|\phi - u\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$, then the problem (1.3) has a unique solution $v \in C^\infty(\Omega)$.*

Let us explicitly note that, if u is a solution of class C_h^2 of (1.3), then it is of class C^∞ . Using mollifiers we can mimic the proof of [8, Theorem 1.2] and we can easily conclude the following

Lemma 4.2. *Let $G_u = \{z = u(x) : x \in \Omega\}$ be a C_h^2 graph in \mathbb{H}^n with mean curvature $H_u = g \in C^\infty(\Omega)$, $n > 1$. Then u is a smooth function.*

Proof. We denote by u_ε the standard mollifiers

$$u_\varepsilon(x) = \int_{\mathbb{R}^{2n}} u(y) \varphi_\varepsilon(x - y) \varepsilon^{-n} dy.$$

It is simple to verify that

- (i) $X_{\bar{u}} f_\varepsilon \rightarrow X_{\bar{u}} f$ uniformly on compact subsets of Ω , for $\varepsilon \rightarrow 0$.
- (ii) $X_{\bar{u}}^2 f_\varepsilon \rightarrow X_{\bar{u}}^2 f$ uniformly on compact subsets of Ω , for $\varepsilon \rightarrow 0$.

For every ε , the function u_ε satisfies the representation formula (4.2) in [8]. Letting ε go to 0, the same formula is satisfied by u . Hence we can now proceed as in [8, Section 4], and obtain the smoothness result. \square

Finding a solution of the problem (1.3) is equivalent to prove the invertibility of the map

$$F : C_d^{2,\alpha}(\Omega) \rightarrow C_d^{0,\alpha}(\Omega) \times C(\partial\Omega)$$

defined by

$$F(w) = (H_w, w|_{\partial\Omega}),$$

As it is well known, the local invertibility property of F can be studied through its differential $dF(u) = L_u$, so that we focus on this linear operator. The continuity of its inverse is expressed by the Schauder estimates at the boundary, in suitable C^α spaces:

Proposition 4.3. *Let L_u defined by (3.1), where $u : \Omega \subset \rightarrow \mathbb{R}$ is a smooth function. Let $f \in C_d^\alpha(\Omega)$ and let $v \in C_{loc}^{2+\alpha}(\Omega)$ a bounded function satisfying $L_u v = f$ in Ω . Then $v \in C_d^{2+\alpha}(\Omega)$ and there exists $C > 0$ (independent of v) such that*

$$\|v\|_{C_d^{0,\alpha}(\Omega)} \leq C \left(\|v\|_{L^\infty(\Omega)} + \|d^2 f\|_{C_d^{0,\alpha}(\Omega)} \right).$$

The proof of Proposition 4.3 is standard with the techniques developed in the last two decades, since L_u is a sub-elliptic second order linear operator with smooth coefficients. Interior Schauder estimates for L_u can be found in [6]. Here we need similar estimates, but we need to provide an explicit estimate of the constant C in terms of the distance from K to the exterior of Ω .

Due to the existence of a fundamental solution for L_u , it is also possible to obtain these estimates, mimicking the classical argument presented in the Euclidean setting in [22, § 4].

Proof of Lemma 4.1. We consider the map $F : C_d^{2,\alpha}(\Omega) \rightarrow C_d^{0,\alpha}(\Omega) \times C(\partial\Omega)$ defined by

$$F(w) = (H_w, w|_{\partial\Omega}),$$

where H_w is the mean curvature of the graph G_w . The differential of F in u is

$$dF_u(v) = (L_u v, v|_{\partial\Omega}).$$

The kernel of dF_u does not contain a non-trivial function, since otherwise the first eigenvalue of L_u in Ω would be smaller than or equal to 0. On the other hand, the problem

$$\begin{cases} L_u v = f, & \text{in } \Omega \\ v = \phi, & \text{in } \partial\Omega \end{cases}$$

has a solution for all $f \in C_d^{0,\alpha}(\Omega)$ and $\phi \in C(\Omega)$. From Proposition 4.3 the inverse of dF_u is continuous. By the Implicit Function Theorem, there is a diffeomorphism from a neighborhood of u in $C_d^{2,\alpha}(\Omega)$ into a neighborhood of $(H_u, u|_{\partial\Omega})$ in $C_d^{0,\alpha}(\Omega) \times C(\partial\Omega)$. In particular, there exists

$\varepsilon_0 > 0$ so that, for every ϕ such that $\|\phi - u\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$, the given problem has a solution v such that the graph G_v of v is area-stationary with zero mean curvature and $G_v|_{\partial\Omega} = \psi$.

Finally the function v is of class C^∞ for Corollary 4.2. \square

5. AREA MINIZING PROPERTY

Let us note that, if u is strictly stable, then the first eigenvalue $\lambda_1(\Omega)$ of the operator L_u is positive, in fact if λ is an eigenvalue of L_u with eigenfunction $v \in C_h^2(\Omega) \cap C_0(\Omega)$ we have

$$0 < - \int_{\Omega} v L_u v \, d\mathcal{L}^{2n} = \lambda \int_{\Omega} v^2 \, d\mathcal{L}^{2n}.$$

As a consequence the following Maximum principle holds:

Lemma 5.1. *We consider the operator $L_u v$ defined in (3.1). Suppose that $\lambda_1(\Omega) > 0$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of L on a bounded $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^n$. Assume that $Lv \leq 0$ and $\inf_{\partial\Omega} v > 0$, then $\inf_{\Omega} v > 0$.*

Proof. First we observe that the minimum of v can not be zero, otherwise $v \equiv 0$. This can be achieved mimic the classical proof of the Hopf's maximum principle (see for example [22, Theorem 3.5] or [26, § 3]) and replacing the role of Euclidean balls with the sets $\delta_r O_1$, $r > 0$, introduced in [5, p. 1161].

Now we suppose that v achieved a negative minimum in the interior of Ω , then we can extend the the operator L_u in a small neighborhood $\tilde{\Omega}$ of Ω and we can suppose that the first eigenvalue $\lambda_1(\tilde{\Omega})$ is positive as well as the restriction to Ω of the eigenfunction v_1 associated to $\lambda_1(\tilde{\Omega})$. We consider the function $w = v/v_1$ on Ω , that must has a negative minimum in some interior point p . Hence at the point p

$$0 \geq L_u(v) = L_u(wv_1) = L_u(v_1)w + v_1 \sum_{i,j=1}^{2n-1} X_i^u(a_{ij}(\nabla^u u)X_j^u w) \geq -\lambda_1(\tilde{\Omega})v_1 w$$

and we can conclude that $w(p) \geq 0$ and $v(p) \geq 0$ in contradiction with our assumption. \square

Lemma 5.2. *Let $u \in Lip(\tilde{\Omega}) \cap C_h^2(\Omega)$ and let G_u be a minimal graph in \mathbb{H}^n , $n > 1$. We assume that G_u is strictly stable in a domain $\Omega \subset \tilde{\Omega}$. Then there exist $\varepsilon_0 > 0$, a tubular neighborhood U of G_u and a family G_{u_ε} of surfaces with vanishing mean curvature such that $G_{u_0} = G_u$ and G_{u_ε} is a foliation of U for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Proof. As before we consider the map

$$F(w) = (H_w, w|_{\partial\Omega}).$$

We have proved that there is a diffeomorphism from a neighborhood of u in $C_d^{2,\alpha}(\Omega)$ into a neighborhood of $(H_u, u|_{\partial\Omega})$ in $C_d^{0,\alpha}(\Omega) \times C(\partial\Omega)$. In particular, there exists $\varepsilon_0 > 0$ so that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there is a function u_ε such that the graph G_{u_ε} of u_ε is area-stationary with zero mean curvature and $G_{u_\varepsilon}|_{\partial\Omega} = u|_{\partial\Omega} + \varepsilon$.

Let us check that the union of the graphs G_{u_ε} provide a foliation of a tubular neighborhood of G_u in \mathbb{H}^n . The graphs G_{u_ε} provide a variation of G_u in \mathbb{H}^n . If we compute the variational function $v := \frac{d}{d\varepsilon}|_{\varepsilon=0} u_\varepsilon$, then v satisfies

$$\begin{cases} L_u v = 0, & \text{in } \Omega \\ v = 1, & \text{in } \partial\Omega \end{cases}.$$

Since by Lemma 5.1 below $v > 0$, choosing ε_0 smaller if necessary, we would obtain that the graph G_{u_ε} , for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ foliate a tubular neighborhood of G_u . □

Now we are ready to prove the main result of the paper

Theorem 5.3. *Let $u \in Lip(\tilde{\Omega}) \cap C_h^2(\Omega)$ and let G_u be a minimal graph in \mathbb{H}^n , $n > 1$. We assume that G_u is strictly stable in a domain $\Omega \subset \tilde{\Omega}$. Then there exists a tubular neighborhood U of Ω such that for any C_h^1 graph $S \subset U$, $\partial\Omega = \partial S$, we have $A(G_u) \leq A(S)$ or $G_u = S$.*

Proof. From Lemma 5.2 we know the existence of a family $\{G_{u_\varepsilon}\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ of minimal graphs G_{u_ε} that foliate a tubular neighborhood U of G_u . Let $v : \Omega \rightarrow \mathbb{R}$ be a $C_h^1(\Omega)$ function such that $u \equiv v$ on $\partial\Omega$. We denote by E the region bounded by $G_u \cup G_v$. Using the Gauss-Green formula, [18, Theorem 2.3], and the fact that $\langle (v_h)_{G_{u_\varepsilon}}, (v_h)_{G_u} \rangle = 1$ and $\langle (v_h)_{G_{u_\varepsilon}}, (v_h)_{G_v} \rangle \geq -1$, we get

$$\begin{aligned} 0 &= \int_E \operatorname{div}_{\mathbb{H}^n}((v_h)_{\Sigma_\varepsilon}) d\mathcal{L}^{2n+1} = \int_{\mathbb{H}^n} \langle (v_h)_{G_{u_\varepsilon}}, (v_h)_{G_u} \rangle dP(G_u) + \int_{\mathbb{H}^n} \langle (v_h)_{G_{u_\varepsilon}}, (v_h)_{G_v} \rangle dP(G_v) \\ &\geq A(G_u \cap U) - A(G_v \cap U), \end{aligned}$$

with strict inequality unless $(v_h)_{G_v}$ coincides with $(v_h)_{G_u}$ in all points. Here $(v_h)_{G_u}$ denotes the horizontal projection of the Riemannian unit normal of G_u . We conclude that G_u and G_v coincides by [3, Corollary 1.3]. □

As a direct consequence of this result and the fact that 0 is a stable solution, we obtain

Corollary 5.4. *Then there exist $\varepsilon_0 > 0$, such that if $\|\phi\|_{L^\infty(\partial\Omega)} \leq \varepsilon_0$, then the problem*

$$\begin{cases} H_v = 0, & \text{in } \Omega \\ v = \phi, & \text{in } \partial\Omega \end{cases}$$

has an unique solution $v \in C^\infty(\Omega) \cap C(\bar{\Omega})$, which is a stable minimum for the area functional.

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